ON ABSOLUTE CONVERGENCE OF FOURIER INTEGRALS

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ABSTRACT. New sufficient conditions for representation of a function via the absolutely convergent Fourier integral are obtained in the paper. In the main result, Theorem 1.1, this is controlled by the behavior near infinity of both the function and its derivative. This result is extended to any dimension $d \geq 2$.

1. INTRODUCTION

The possibility to represent a function via the absolutely convergent Fourier integral was studied by many mathematicians and is of importance in various problems of analysis. New sufficient conditions of such type in $\mathbb{R}^d$, $d \geq 1$, are obtained in the paper.

1.1. **Motivation.** To illustrate the importance of such theorems, let us start with a typical example of their application. For simplicity, let us restrict ourselves with the one-dimensional case.

Given algebraic polynomials P_1, P_2 and Q, when, for $D = \frac{d}{dx}$, the inequality

$$||Q(D)f||_{L_q(\mathbb{R})} \leq \gamma(||P_1(D)f||_{L_{p_1}(\mathbb{R})} + ||P_2(D)f||_{L_{p_2}(\mathbb{R})}) \tag{1.1}$$

holds true with a constant γ independent of the function f?

Let us simplify the situation even more by putting only one operator on the right. The initial problem is now reduced to when

$$||Q(D)f||_{L_q(\mathbb{R})} \leq ||P(D)f||_{L_p(\mathbb{R})} \quad ? \tag{1.2}$$

Clearly, there should be $s = \deg Q \leq r = \deg P$. Considering all the functions for which the right-hand side of (1.2) is finite, we deduce that all the solutions of the equation $P(D)f) = 0$ are the solutions of the equation $Q(D)f = 0$ as well, that is, $Q = cP$ with some constant c.

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The problem becomes meaty under assumption $f \in W_p^r(\mathbb{R})$. The latter is a usual notation for the Sobolev space. Necessary conditions for fulfillment of (1.2) in this case are $q \ge p$ when $s < r$, $q = p$ when $s = r$, and when either $q \ne \infty$ or $p \ne 1$

$$\sup_{x\in\mathbb{R}} |\phi(x)| < \infty, \qquad \phi(x) = \frac{Q(ix)}{P(ix)}. \tag{1.3}$$

We have for $|x| \to \infty$, provided (1.3) holds true,

$$\phi(x) = a_0 + \frac{a_1}{x} + O\Big(\frac{1}{x^2}\Big), \quad \phi'(x) = -\frac{a_1}{x^2} + O\Big(\frac{1}{x^3}\Big),$$

and each of the theorems of Section 2 below yields that the Fourier transform $\widehat{\phi - a_0} \in L_1(\mathbb{R})$, or in other words $\phi - a_0$ is the Fourier transform of a function $g \in L_1(\mathbb{R})$.

And if, for example, $f \in W_1^r(\mathbb{R})$, then $\widehat{f^{(k)}}(y) = (iy)^k \widehat{f}(y)$, $0 \le k \le r$, and

$$\widehat{P(D)f}(y) = P(iy)\widehat{f}(y), \qquad \widehat{Q(D)f}(y) = Q(iy)\widehat{f}(y),$$

$$\widehat{Q(D)f} = \phi\, \widehat{P(D)f}.$$

By this we get that $Q(D)f$ is representable as the convolution of g and $P(D)f$ (for general theory of multipliers, see [14, 16, 15]; see also the beginning of Section 2).

In fact, g is also bounded almost everywhere (a.e.). Applying the Young inequality for convolutions (see, e.g., [14, App.A]), we obtain the inequality (1.2) in the general case (for details, see [20], where three criteria for existence of such inequalities are found: on the axis, on the half-axis, and on the circle).

We are now in a position to return to the case of two operators $P_1(D)$ and $P_2(D)$ on the right-hand side of (1.1).

We suppose that $r_2 = \deg P_2 \le \deg P_1 = r_1$, $P_1(x) = I(x)I_1(x)\widetilde{P_1}(x)$, and $P_2(x) = I(x)I_2(x)\widetilde{P_2}(x)$, where the polynomials $\tilde{P}_1$ and $\tilde{P}_2$ never vanish on the imaginary axis $i\mathbb{R}$, while zeros of I, I_1 and I_2, if exist, are located just on $i\mathbb{R}$. Suppose also that I_1 and I_2 have no common zeros. In this case the polynomial Q is divisible by I as well.

Obviously, all the values I_1 takes on $i\mathbb{R}$ are on the same line passing through the origin (as well as those of I_2), hence one may assume that I_1 and I_2 take on $i\mathbb{R}$ only real value (maybe being multiplied by a constant).

If $I_2(ix_1) \neq 0$ for $x_1 \in \mathbb{R}$, and $k = \deg \widetilde{P_1}$, we set

$$P_0(x) = \pm i H(x) I_1(x) + I_2(x), \qquad H(x) = (ix + x_1)^k,$$

where the sign $+$ or $-$ is chosen in such a way that $\deg P_0 I = \deg P$. Then $P_0(ix) \neq 0$ for $x \in \mathbb{R}$. Applying (1.2) three times, we obtain

$$\begin{aligned} ||Q(D)f|| &\leq \gamma_1 ||I P_0(D) f|| \leq \gamma_1 (||H\, I\, I_1(D) f|| + ||I\, I_2(D) f||) \\ &\leq \gamma_2 (||P_1(D) f|| + ||P_2(D) f||). \end{aligned}$$

We mention that similar arguments are applicable for functions of several variables as well; more precisely, for elliptic differential operators and those related. For this, see [14, 4, 1] and references therein.

By the way, (1.2) for $P_1(x) = x^r$, $r \geq 2$, $P_2(x) \equiv 1$, and $Q(x) = x^k$, $1 \leq k \leq r-1$, yields, for example for $f \in W_\infty^r$,

$$||f^{(k)}||_\infty \leq \gamma_3(k,r)\,(||f^{(r)}||_\infty + ||f||_\infty).$$

Substituting εx, with $\varepsilon > 0$, for x, dividing by ε^k and minimizing the right-hand side over $\varepsilon \in (0,\infty)$, we obtain the known multiplier inequality for intermediate derivatives

$$||f^{(k)}||_\infty \leq \gamma_4(k,r) ||f||_\infty^{\frac{k}{r}}\, ||f^{(r)}||_\infty^{1-\frac{k}{r}}.$$

Prior to formulating main results we explain how the paper is organized and fix certain notation and conventions. First, if

$$f(y) = \int\limits_{\mathbb{R}^d} g(x) e^{i(x,y)} dx, \qquad g \in L_1(\mathbb{R}^d),$$

we write $f \in A(\mathbb{R}^d)$, with $||f||_A = ||g||_{L_1(\mathbb{R}^d)}$. In Section 2, known results on representability of a function as the absolutely convergent Fourier integral are given and compared. The proofs of the new results are given in Section 3.

We shall denote absolute constants by c or maybe by c with various subscripts, like c_1, c_2, etc., while $\gamma(...)$ will denote positive quantities depending only on the arguments indicated in the parentheses. We shall also use the notation $\int_{\to 0}$ to indicate that the integral is understood as improper in a neighborhood of the origin, that is, as $\lim\limits_{\delta \to 0+} \int_\delta$.

1.2. **Main results.** We start with the case $d = 1$.

Let $f \in C_0(\mathbb{R})$, that is, $f \in C(\mathbb{R})$ and $\lim f(t) = 0$ as $|t| \to \infty$, and let f be locally absolutely continuous on $\mathbb{R} \setminus \{0\}$.

Theorem 1.1. *Let* $f_0(t) = \sup_{|s|\ge|t|} |f(s)|$.

a) *Let* f' *be bounded a.e. out of any neighborhood of zero and* $f_1(t) = \operatorname{ess\,sup}_{|s|\ge|t|>0} |f'(s)|$. *If, in addition,*

$$A_0 = \int_1^\infty \frac{f_0(t)}{t}\,dt < \infty, \qquad A_1 = \int_0^1 f_1(t) \ln\frac{2}{t}\,dt < \infty$$

and

$$A_{01} = \int_1^\infty \left(\int_t^\infty f_0(s) f_1(s)\,ds\right)^{\frac12} \frac{dt}{t} < \infty,$$

then $f \in A(\mathbb{R})$, *with* $||f||_A \le c(A_0 + A_1 + A_{01})$.

b) *Let* f' *be not bounded near infinity,* $f_\infty(t) = \operatorname{ess\,sup}_{0<|s|\le|t|} |f'(s)|$ *and* $f(t) = 0$ *when* $|t| \le 2\pi$, *with* $f_\infty(4\pi) > 0$. *If, in addition, there exists* $\delta \in (0,1)$ *such that*

$$A_\delta^{1+\delta} = \sup_{t\ge 2\pi} t\, f_0^\delta(t)\, f_\infty(t + 2\pi) < \infty,$$

then $f \in A(\mathbb{R})$ *and* $||f||_A \le \gamma(\delta) A_\delta (1 + A_\delta^{\frac1\delta}(f_\infty(4\pi))^{-\frac1\delta})$.

Conditions of this theorem differ from known sufficient conditions in the way that near infinity combined behavior of both the function and its derivative comes into play (see also the corollary below). Conditions for f_0 near infinity and f_1 near the origin in **a)** are also necessary. For instance, this is the case when $f(t) = 0$ for $t \le 0$, $f \in C^1(0, +\infty)$ and piecewise convex on $[0, \infty)$, since for such functions both conditions are equivalent to convergence of the integral $\int_0^\infty t^{-1} f(t)\,dt$ (see necessary conditions in Section 2). As for the condition $A_{01} < \infty$, it holds, for example, if $(\ln t)^{2+\delta} f_0(t) f_1(t) \in L_1[1, \infty)$ for some $\delta > 0$, but nit for $\delta = 0$.

We note that in **b)** the function can be considered on the whole axis. It should satisfy the same condition A_1 as in **a)** near the origin. We omit this for simplicity.

Corollary 1.2. *If $A_1 < \infty$, $f(t) = O(|t|^{-\alpha})$ for some $\alpha > 0$ and $f'(t) = O(|t|^{-\beta})$ for some $\beta \in \mathbb{R}$ as $|t| \to \infty$, with $\alpha + \beta > 1$, then $f \in A(\mathbb{R})$. If $\alpha + \beta < 1$ such an assertion cannot be valid.*

Let now $f : \mathbb{R}^d \to \mathbb{C}$ with $d \ge 2$. We will give a direct generalization of **a)** in Theorem 1.1 to higher dimensions.

To formulate a multivariate extension of that result, we introduce certain notation. Let χ, η and ζ be d-dimensional vectors with the entries either 0 or 1 only. Each of these vectors or even two of them can be zero vectors $\mathbf{0} = (0, 0, ..., 0)$. The inequality of vectors means the same inequality for all pairs of their corresponding components.

Similarly to the one-dimensional case, we set for $\eta + \zeta = \mathbf{1} = \{1, 1, ..., 1\}$.

$$f_{\eta,\zeta}(x) = \sup_{\substack{|u_i| \ge |x_i|, \\ i:\eta_i=1}} \ \operatorname*{ess\,sup}_{\substack{|u_j| \ge |x_j| > 0, \\ j:\zeta_j=1}} |D^{\zeta} f(u)|,$$

where

$$D^{\chi} f(x) = \left(\prod_{j:\chi_j=1} \frac{\partial}{\partial x_j} \right) f(x).$$

We denote by $\mathbb{R}_\zeta$ the Euclidean space of dimension $\zeta_1 + ... + \zeta_d$ with respect to the variables x_j with js for which $\zeta_j = 1$; correspondingly x_ζ is an element of this space.

Theorem 1.3. *Let $f \in C_0(\mathbb{R}^d)$ and let f and its partial derivatives $D^\eta f$, $\mathbf{0} \le \eta < \mathbf{1}$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let also partial derivatives $D^\eta f$, $\mathbf{0} < \eta \le \mathbf{1}$ be almost everywhere bounded out of any neighborhood of each coordinate hyperplane. If*

$$A_{\chi,\eta,\zeta} = \int_0^1 ... \int_0^1 \prod_{k:\zeta_k=1} \ln(2/x_k)\, dx_k$$

$$\int_1^\infty ... \int_1^\infty \Bigg(\int\limits_{\prod_{j:\eta_j=1} [u_j,\infty)} f_{\chi+\eta,\zeta}(x) f_{\chi,\eta+\zeta}(x)\, dx_\eta \Bigg)^{1/2} \prod_{\substack{i:\chi_i=1 \\ \text{or } \eta_i=1}} \frac{du_i}{u_i} < \infty \tag{1.4}$$

for all χ, η and ζ such that $\chi+\eta+\zeta = \mathbf{1} = \{1, 1, ..., 1\}$, then $f \in A(\mathbb{R}^d)$.

The authors understand, of course, that (1.4) is a (quite large) number of conditions not easily observable, in a sense. To clarify this point, we give the two-dimensional version of this theorem. We will use separate letter for each variable rather than subscripts; also no need in using vectors for defining analogs of f_0 and f_1 - we just denote them by using subscripts 0 or 1 to indicate majorizing in the corresponding variable: f_{00}, f_{01}, f_{10} and f_{11}.

Theorem 1.3′. *Let $f(x,y)\in C_0(\mathbb{R}^2)$ and let f and its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ be locally absolutely continuous on $(\mathbb{R}\setminus\{0\})^2$ in each variable. Let also partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x\partial y}$ be almost everywhere bounded out of any neighborhood of each coordinate axis. If*

$$\int\limits_0^1\int\limits_0^1 f_{11}(x,y)\ln\frac{2}{x}\ln\frac{2}{x}\,dx\,dy<\infty,\qquad \int\limits_1^\infty\int\limits_1^\infty \frac{f_{00}(x,y)}{xy}\,dx\,dy<\infty,$$

$$\int\limits_1^\infty\int\limits_1^\infty\Big(\int\limits_x^\infty\int\limits_y^\infty f_{00}(s,t)f_{11}(s,t)\,ds\,dt\Big)^{1/2}\frac{dx}{x}\frac{dy}{y}<\infty,$$

$$\int\limits_0^1\int\limits_1^\infty f_{10}(x,y)\ln\frac{2}{x}\,dx\,\frac{dy}{y}<\infty,$$

$$\int\limits_0^1\int\limits_1^\infty\Big(\int\limits_y^\infty f_{10}(x,t)f_{11}(x,t)\,dt\Big)^{1/2}\ln\frac{2}{x}\,dx\,\frac{dy}{y}<\infty,$$

$$\int\limits_1^\infty\int\limits_0^1 f_{01}(x,y)\ln\frac{2}{y}\,\frac{dx}{x}\,dy<\infty,$$

$$\int\limits_1^\infty\int\limits_0^1\Big(\int\limits_x^\infty f_{01}(s,y)f_{11}(s,y)\,ds\Big)^{1/2}\ln\frac{2}{y}\,\frac{dx}{x}\,dy<\infty,$$

$$\int\limits_1^\infty\int\limits_1^\infty\Big(\int\limits_y^\infty f_{00}(x,t)f_{01}(x,t)\,dt\Big)^{1/2}\frac{dx}{x}\,\frac{dy}{y}<\infty,$$

and

$$\int_1^\infty \int_1^\infty \left(\int_x^\infty f_{00}(s,y) f_{10}(s,y)\, ds \right)^{1/2} \frac{dx}{x} \frac{dy}{y} < \infty,$$

then $f \in A(\mathbb{R}^2)$.

As is mentioned, already for $d = 3$ no way to briefly write down all the conditions. Let is give only one of them, quite typical and completely "mixed":

$$\int_0^1 \int_1^\infty \int_1^\infty \left(\int_z^\infty f_{100}(x,y,u) f_{101}(x,y,u)\, du \right)^{1/2} dx\, \frac{dy}{y} \frac{dz}{z} < \infty;$$

here f_{100} and f_{101} is a clear analog of the above notation.

2. Known results

Let $\phi : \mathbb{R}^d \to \mathbb{C}$ be a bounded measurable function. We define on $L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ a linear operator Φ via the following equality for the Fourier transform of a function $f \in L_2 \cap L_p$

$$\widehat{\Phi f}(y) = \phi(y) \widehat{f}(y).$$

Clearly, $\Phi f \in L_2(\mathbb{R}^d)$, and if there exists a constant D such that for all $f \in L_2 \cap L_p(\mathbb{R}^d)$

$$||\Phi(f)||_{L_p} \le D ||f||_{L_p},$$

then the operator Φ is called the Fourier multiplier from L_p into $L_p(\mathbb{R}^d)$ (written $\phi \in M_p(\mathbb{R}^d)$), whence $||\Phi||_{L_p \to L_p} = \inf D$.

Sufficient conditions for a function to be a multiplier in L_p spaces with $1 < p < \infty$ for both multiple Fourier series and Fourier integrals were studied by Marcinkiewicz, Mikhlin, Hörmander, Lizorkin, and others (see, e.g., [14, Ch.4] and [15]). There holds $M_1 = M_\infty \subset M_p$, $1 < p < \infty$. When $p = 1$ and $p = \infty$ each Fourier multiplier is the convolution of the function f and a finite (complex-valued) Borel measure on $\mathbb{R}^d$:

$$\Phi f(x) = \int_{\mathbb{R}^d} f(x-y) d\mu(y), \qquad ||\Phi||_{L_1 \to L_1} = ||\Phi||_{L_\infty \to L_\infty} = \mathrm{var}\mu,$$

while $\phi \in M_1 = M_\infty$ iff $\phi \in B(\mathbb{R}^d)$, where

$$B(\mathbb{R}^d) = \{\phi : \phi(y) = \int\limits_{\mathbb{R}^d} e^{i(x,y)} d\mu(x), \qquad ||\phi||_B = \mathrm{var}\mu < \infty\}$$

(see, e.g., [16, Ch.1]). If the measure μ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$, then we write $\phi \in A(\mathbb{R}^d)$, where

$$\begin{aligned} A(\mathbb{R}^d) &= \{\phi : \phi(y) = (2\pi)^{\frac{d}{2}} \widehat{g}(-y) = \int\limits_{\mathbb{R}^d} e^{i(x,y)} g(x) dx, \\ ||\phi||_A &= \int\limits_{\mathbb{R}^d} |g(x)| dx < \infty\}. \end{aligned}$$

The space $B(\mathbb{R}^d)$ is the Banach algebra with respect to pointwise multiplication, while $A(\mathbb{R}^d)$ is an ideal in $B(\mathbb{R}^d)$. As is known, the two algebras are locally geared in the same way, thus the difference between A and B is revealed in the behavior of functions near infinity. We also note that if $\phi \in B(\mathbb{R}^d)$, $\lim \phi(y) = 0$ as $|x| \to \infty$ and ϕ is of finite total Vitali variation off a cube, then $\phi \in A(\mathbb{R}^d)$ ([19, Theorem 2]).

We remind the reader that total Vitali variation of the function $\phi : E \to \mathbb{C}$, with $E \subset \mathbb{R}^d$, is defined as follows. If $\{e_j^0\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$, and the boundary of E consists of a finite number of planes given by equations $x_j = c_j$, then

$$V(f) = \sup \sum |\Delta_u f(x)|, \qquad \Delta_u f(x) = (\prod_{j=1}^d \Delta_{u_j}) f(x),$$

where $u = (u_1, \cdots, u_d)$ and

$$\Delta_{u_j} f(x) = f(x + u_j e_j^0) - f(x - u_j e_j^0), \quad 1 \le j \le d. \tag{2.1}$$

Here Δ_u is the mixed difference with respect to the vertices of the parallelepiped $[x-u, x+u]$ and sup is taken over any number of non-overlapping parallelepipeds in E. For smooth enough functions on E such as indicated above, one has

$$V(f) = \int\limits_E \left| \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \right| dx.$$

We note that in Marcinkiewicz's sufficient condition for M_p, $1 < p < \infty$, only the finiteness of total variations over all dyadic parallelepipeds with no intersections with coordinate hyper-planes is assumed (see, e.g., [14]).

Many mathematicians studied the properties of absolutely convergent Fourier series rather than integrals, starting from one paper by S.N. Bernstein (see, e.g., [9]; for multidimensional results see, e.g., [18]). Various sufficient conditions for absolute convergence of Fourier integrals were obtained by Titchmarsh, Beurling, Karleman, Sz.-Nagy, Stein, and many others. One can find more or less comprehensive and very useful survey on this problem in [13], with 65 bibliographical references therein.

Let us give some results not contained in that survey as well as relations between them and other results of such type. The other reason for giving these is that some of these results will essentially be used in proofs.

Pólya proved that each even, convex and monotone decreasing to zero function on $[0, \infty)$ belongs to $A(\mathbb{R})$. In fact, such function belongs even to $A^*(\mathbb{R})$, that is, not only $\widehat{f} \in L_1(\mathbb{R})$, but also $\sup_{|s|\ge|t|} |\widehat{f}(s)| \in L_1(\mathbb{R})$ (see [19] or [21]). By this, f may decrease arbitrarily slowly. What is really important, as Pólya observed, is that $\widehat{f}(y) \ge 0$.

Zygmund proved that if an odd function f is compactly supported and convex in a right neighborhood of the origin, it admits an extension to $A(\mathbb{R})$ iff the improper integral $\int_{\to 0} t^{-1} f(t)\,dt$ converges (see [9]). There is a more general statement (Lemma 6 in [19]): if $f \in C_0(\mathbb{R})$ and piece-wise convex, then for any $y \ne 0$

$$|\widehat{f}(y)| \le \gamma(q)\omega(f; \frac{\pi}{|y|}),$$

where $\omega(f; h)$ is the modulus of continuity and q is the number of intervals on each of them $\mathrm{Re} f$ and $\mathrm{Im} f$ are either convex or concave. It follows from this that if a function is also odd and on any interval not containing zero satisfies the $\mathrm{Lip}\alpha$, $\alpha > 0$, condition, then $f \in A(\mathbb{R})$ iff the integral $\int_0^\infty t^{-1} f(t)\,dt$ converges. Therefore, if f is odd and $f(x) \ge 0$ for $x \ge 0$, then $f \in A(\mathbb{R})$ does not yield $f \in L_1(\mathbb{R})$ (cf. Theorem 2.8 in [13]).

For a real, bounded and locally absolutely continuous function to be the difference of two bounded convex functions on $[0, \infty)$, (quasi-convex), it is necessary and sufficient that

$$\int_0^\infty t|df'(t)| < \infty.$$

A similar fact is well known for sequences.

In the paper by Beurling [2] more general condition was given:

$$V^*(f) = \int_0^\infty \operatorname*{ess\,sup}_{s\ge t} |f'(s)|\,dt < \infty.$$

This condition is less restrictive that that of convexity(and quasi-convex), but more severe than the finiteness of the total variation.

If, in addition, $f \in C_0[0,\infty)$ and $f(t) = 0$ for $t < 0$, then for each $y \in \mathbb{R} \setminus \{0\}$

$$\widehat{f}(y) \;=\; \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)e^{-iyt}dt = -\frac{i}{y\sqrt{2\pi}}f(\frac{\pi}{2|y|}) + \theta F(y),$$

with $|\theta| \le c$ and $||F||_{L_1(\mathbb{R})} \le V^*(f)$.

This can be found in [21, 6.4.7b and 6.5.9]. It is interesting that for f convex $F(y)$ can be considered monotone decreasing as $|y|$ increases, while it is not the case for the class in question. The point is that integrability of the monotone majorant of $|\widehat{f}|$ is related to summability of Fourier series at Lebesgue points (see [21, 8.1.3]). Monotonicity of F in the latter case will then lead to coincidence of results for V^* and convex functions, which is impossible.

Theorem A ([2]). Let $f \in C_0(\mathbb{R})$ and there exists a function g such that

$$|f(t) - f(t+h)| \le |g(t) - g(t+h)| \qquad (t, h \in \mathbb{R})$$

and $g \in A^*(\mathbb{R})$, that is, $g = \widehat{\psi}$, with $\psi^*(t) =\operatorname{ess\,sup}_{|t|\ge|s|} |\psi(s)| \in L(\mathbb{R})$. Then $f \in A(\mathbb{R})$.

For general properties of the algebra $A^*(\mathbb{R})$, see [3].

On the other hand, many works were devoted to the related question of the boundedness and asymptotics of L_1-norms over the period of the sequence of periodic functions against their Fourier coefficients (see, e.g., [17], and also [21, 7.2.8, 8.1.1] and [11]).

Let us now give F. Riesz's criterion of the absolute convergence of Fourier integrals (its counterpart for series can be found in [9]).

Theorem B. *Function $f \in A(\mathbb{R}^d)$ if and only if it is representable as the convolution of two functions from $L_2(\mathbb{R}^d)$:*

$$f(x) = \int_{\mathbb{R}^d} f_1(y) f_2(x-y)\,dy, \qquad f_1, f_2 \in L_2(\mathbb{R}^d). \tag{2.2}$$

By this, $||f||_A \le ||f_1||_2\, ||f_2||_2$.

Proof. The proof is based on the unitarity of the Fourier operator in $L_2(\mathbb{R}^d)$. With (2.2) in hand,

$$f(x) = \int_{\mathbb{R}^d} \widehat{f_1}(u) \overline{\widehat{f_2(x-u)}}\,du = \int_{\mathbb{R}^d} \widehat{f_1}(u)\, \overline{\widehat{f_2}(-u)}\, e^{i(x,u)} du$$

and by the Cauchy-Schwarz-Bunyakovskii inequality

$$||f||_A = \int_{\mathbb{R}^d} |\widehat{f_1}(u)|\, |\widehat{f_2}(-u)|\, du \le ||\widehat{f_1}||_2\, ||\widehat{f_2}||_2 = ||f_1||_2\, ||f_2||_2.$$

Farther, if

$$f(x) = \int_{\mathbb{R}^d} g(y) e^{i(x,y)} dy, \qquad ||g||_1 = \int_{\mathbb{R}^d} |g(y)|\, dy < \infty,$$

then

$$g(y) = |g(y)|^{\frac{1}{2}}\, (|g(y)|^{\frac{1}{2}} \operatorname{sign} g(y)),$$

where, as usual, $\operatorname{sign} 0 = 0$ and $\operatorname{sign} z = \frac{z}{|z|}$ when $z \ne 0$, and each of the two factors on the right, as well as their Fourier transforms, belongs to $L_2(\mathbb{R}^d)$. It remains to apply the above given formulae for convolution in the reverse order. □

Let us demonstrate how to derive effective sufficient conditions from this criterion.

Assume $f \in C(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $(-\Delta)^{\frac{\alpha}{2}} f \in L_2(\mathbb{R}^d)$, where Δ is the Laplace operator. Taking into account that

$$\widehat{(-\Delta)^{\alpha/2}} f(y) = |y|^\alpha \widehat{f}(y) \in L_2(\mathbb{R}^d),$$

and for $\alpha > d/2$

$$\int\limits_{\mathbb{R}^d} \frac{dy}{(1+|y|)^2} = \gamma(\alpha) \int\limits_0^\infty \frac{t^{\alpha-1}}{(1+t^\alpha)^2} dt < \infty,$$

we get the product of the two functions from $L_2(\mathbb{R}^d)$

$$\widehat{f}(y) = (\widehat{f}(y)(1+|y|^\alpha)) \frac{1}{1+|y|^\alpha}.$$

Therefore $f \in A(\mathbb{R}^d)$ when $\alpha > d/2$.

Another differential operators can be used in the same way, say elliptic, while applying embedding theorems allows one to digress on the function classes defined via moduli of continuity of partial derivatives.

There is one more criterion (approximative) from which in [21, 6.4.3], for example, known sufficient conditions with different smoothness in various variables are derived.

Let us go on to necessary conditions for dimension one. Obviously, for each $\lambda \in \mathbb{R}$

$$||\overline{f}||_A = ||f(\lambda \cdot)||_A = ||f(\cdot + \lambda)||_A = ||e^{i\lambda(\cdot)} f(\cdot)||_A = ||f||_A.$$

let $f \in A(\mathbb{R})$ and $f = \sqrt{2\pi}\widehat{g}$, where $g \in L_1(\mathbb{R})$. Then the trigonometrically conjugate function (the Hilbert transform) is

$$\begin{aligned}
\widetilde{f}(x) &= \frac{1}{\pi} \int\limits_{\to 0}^{\to\infty} \frac{f(x+t) - f(x-t)}{t} dt \\
&:= \lim_{\varepsilon \to +0, M \to +\infty} \frac{1}{\pi} \int\limits_\varepsilon^M \frac{f(x+t) - f(x-t)}{t} dt \\
&= \lim_{\varepsilon \to +0, M \to +\infty} \frac{1}{\pi} \int\limits_\varepsilon^M dt \int\limits_{-\infty}^{+\infty} g(y) \frac{e^{iy(x+t)} - e^{iy(x-t}}{t} dy \\
&= \frac{2i}{\pi} \lim_{\varepsilon \to +0, M \to +\infty} \int\limits_{-\infty}^{+\infty} g(y) e^{ixy} dy \int\limits_\varepsilon^M \frac{\sin ty}{t} dt.
\end{aligned}$$

Since the absolute values of the integrals over $[\varepsilon, M]$ are bounded by an absolute constant, it is possible to pass to the limit under integral sign. This yields

$$\widetilde{f}(x) = i \int_{-\infty}^{+\infty} g(y) e^{ixy} \mathrm{sign} y dy, \qquad ||\widetilde{f}||_A = ||f||_A.$$

We mention that the improper integral in the definition of $\widetilde{f}$ converges everywhere (and uniformly in x), but not necessarily absolutely.

In [19, Theorem 3], a necessary condition for belonging to $A(\mathbb{R}^d)$ is given. It is valid for both radial and non-radial functions of d variables and depends on dimension d.

To formulate the next result on which much in the proofs of our new results is based on, we remind that $\Delta_u f = \Delta_{u_1,\cdots,u_d} f$ is defined by (2.1).

Theorem C (Lemma 4 in [19]). *Let $f \in C_0(\mathbb{R}^d)$.*

a) *If*

$$\sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_d=-\infty}^{\infty} 2^{\frac{1}{2}\sum_{j=1}^d s_j} ||\Delta_{\frac{\pi}{2^{s_1}},\cdots,\frac{\pi}{2^{s_d}}}(f)||_2 < \infty,$$

where the norm is that in $L_2(\mathbb{R}^d)$, then $f \in A(\mathbb{R}^d)$.

b) *If $f = \widehat{g}$, with $g \in L(\mathbb{R}^d)$, and for $|u_j| \ge |v_j|$ when* $\mathrm{sign} u_j = \mathrm{sign} v_j$ *for all $1 \le j \le d$ there holds $|g(u)| \le |g(v)|$, then the series in* **a)** *converges.*

Let us comment on this result.

The convergence of the series condition in **a)** is of Bernstein type. The second named author has learned from [4] that assertions very similar to **a)** of Theorem C were earlier obtained in [6] and [8] (these results are also commented in [4]).Besides that, the same assertion **a)** is reproved in [4, Theorem 3]; the author had apparently been unaware of existence of [19]. Farther, more general problem with power weights is studied in [12].

However, it is worth mentioning that **b)** is proved in [19] as well. Not only this is a necessary condition for certain subclass, but application of this assertion to extension of Beirling's theorem (see Theorem A above) on functions of any number of variables is given in the same paper [19]. To this extent, mixed differences are used instead of those simple. In turn, the following sufficient condition for belonging to $A(\mathbb{R}^d)$ was derived from that generalized Beurling theorem.

Theorem D ([19] or [21]). *If $f(x) = \int\limits_{\substack{|x_j| \le |u_j|,\\ 1 \le j \le d}} g(u) du$, with*

$$\int\limits_{\mathbb{R}^d} \operatorname*{ess\,sup}_{|u_j|\ge|v_j|,1\le j\le d} |g(u)|\,dv < \infty,$$

then $f \in A(\mathbb{R}^d)$.

As is known, the Hausdorff-Young inequality gives sufficient condition for $\widehat{f} \in L_p$, $p > 2$. In [21, 6.4.2] both **a)** and **b)** of Theorem C are generalized to the case where $\widehat{f} \in L_1 \cap L_p$, $p \in (0, 2)$. We also mention that when $f(x) = 0$ for $x \in \mathbb{R} \setminus [0, \pi]$ and $\widehat{f} \in L_1 \cap L_p(\mathbb{R})$, $p \in (0, 1]$ the following necessary condition holds true (see the same reference):

$$\int\limits_0^\pi (|f(x)|^p + |f(\pi - x)|^p) x^{p-2} dx \le \gamma(p) ||\widehat{f}||_p^p.$$

Let us give a recent simple sufficient condition.

Theorem E ([1, Theorem 1]).

If $f \in C(\mathbb{R}^d)$, for any $\delta = (\delta_1, \cdots, \delta_d)$, with $\delta_j = 0$ or 1, $1 \le j \le d$,

$$\lim_{|x_j|\to\infty} \frac{\partial^{\sum \delta_j} f(x)}{\partial x_1^{\delta_1} \cdots \partial x_d^{\delta_d}} = 0, \qquad 1 \le j \le d,$$

and for some $\varepsilon \in (0, 1)$

$$\left| \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \right| \le \frac{A}{\Pi_{j=1}^d |x_j|^{1-\varepsilon} (1 + |x_j|)^{2\varepsilon}},$$

then $||f||_A \le A\gamma(\varepsilon)$.

This theorem is a simple consequence of Theorem D. To make the paper self-contained, let us present this argument.

Proof of Theorem E. We first mention that integrating over $[x_j, \infty)$ an inequality for the mixed derivative, one obtains a similar inequality for derivatives of smaller order.

Let, for simplicity, $d = 2$. If f as a function of x_1 i x_2 is even in both x_1 and x_2, then

$$f(x_1, x_2) = \int_{|x_1|}^\infty du \int_{|x_2|}^\infty \frac{\partial^2 f(u, v)}{\partial u \partial v}\, dv,$$

and Theorem D is immediately applicable.

Any function f is representable as a sum of at most 4 summands, each of them is a function either even or odd in x_1 and x_2 that satisfies the assumptions of Theorem E.

Let, for example, $f_1(-x_1, x_2) = -f_1(x_1, x_2)$, while $f_1(x_1, -x_2) = f_1(x_1, x_2)$. The function

$$h_\mu(t) = \begin{cases} |t|^\mu \mathrm{sign} t, & -1 \le t \le 1, \\ |t|^{-\mu} \mathrm{sign} t, & |t| > 1, \end{cases}$$

belongs to $A(\mathbb{R})$ for any $\mu > 0$. Clearly, always $A(\mathbb{R}) \subset B(\mathbb{R}^2)$.The function $\frac{f}{h_\mu}$ also satisfies assumptions of Theorem E for $\mu > 0$ small enough. As a function even in both x_1 and x_2 it belongs to $A(\mathbb{R}^2)$ due to above argument. Hence

$$f_1 = \left(\frac{f_1}{h_\mu}\right) h_\mu \in A(\mathbb{R}^2),$$

which completes the proof of Theorem E. □

We mention that there exist results similar to Theorem D for even-odd and just odd functions (general case) in [7].

But even more general asymptotic formulae for the Fourier transform can be found in [10].

In the problems of integrability of the Fourier transform the following T-transform of a function $h(u)$ defined on $(0, \infty)$ is of importance

$$Th(t) = \int_{\to 0}^{t/2} \frac{h(t+s) - h(t-s)}{s} \, ds. \tag{2.3}$$

In [5] it is called the Telyakovskii transform. The reason is that in an important asymptotic result for the Fourier transform [10] (cf. [5]) it is used to generalize Telyakovskii's result for trigonometric series.

It is clear that the T-transform should be related to the Hilbert transform; this is revealed and discussed in [10] and later on in, e.g., [5], [11], etc. In particular, the space that proved to be of importance is, related to the real Hardy space, that of functions with both derivative and T-transform of the derivative being integrable.

We need additional notation different from that in [10] and better, in our opinion. Let us denote by $T_j g(x)$ the T-transform of a function g of multivariate argument with respect to the j-th (single) variable:

$$T_j g(x) = \int\limits_{x_j/2}^{3x_j/2} \frac{g(x)}{x_j - t}\,dt = \int\limits_0^{x_j/2} \frac{g(x-te_j^0) - g(x+te_j^0)}{t}\,dt.$$

Analyzing the proof of Theorem 8 in [10], one can see that this theorem can be written in the following asymptotic form.

Theorem F. *Let f be defined on $\mathbb{R}_+^d$; let all partial derivatives taken one time with respect to each variable involved be locally absolutely continuous with respect to any other variable all of these derivatives vanish at infinity as $x_1 + ... + x_d \to \infty$. Then for any $x_1, ..., x_d > 0$ and for any set of numbers $\{a_j : a_j = 0$ or $1\}$ we have*

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+^d} f(u) \prod_{j=1}^d \cos(x_j u_j - \pi a_j/2)\,du_j \\
&= (-1)^{d-1} f\Big(\frac{\pi}{2x_1}, ..., \frac{\pi}{2x_d}\Big) \prod_{j=1}^d \frac{\sin(\pi a_j/2)}{x_j} \\
&+ \sum_{\substack{\chi+\eta+\zeta=\mathbf{1},\\ \chi\ne\mathbf{0}}} \int\limits_{\mathbb{R}_+^d} \prod_{i:\chi_i\ne 0} \frac{\sin(\pi a_j/2)}{x_j} F_{\eta+\zeta}(x)\,dx,
\end{aligned} \tag{2.4}$$

where $F_{\eta+\zeta}$ functions satisfying

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+^d} |F_{\eta+\zeta}(x)|\,dx \\
&\le c \int\limits_{\mathbb{R}_+^d} \prod_{i:\chi_i\ne 0} \frac{\sin(\pi a_j/2)}{x_j} \Big| \prod_{j:\eta_j\ne 0} T_j \prod_{k:\tau_k\ne 0} D_k f(x) \Big|\,dx.
\end{aligned} \tag{2.5}$$

Integrating the summands in this theorem, we obtain

Corollary 2.1. *Let f be as in* **Theorem F**. *If for all $\chi + \eta + \zeta = \mathbf{1}$, with $\chi \ne \mathbf{0}$, all the values of type* (2.5) *are finite, $F \in L(\mathbb{R}^d)$ if and only if*

$$\int\limits_{\mathbb{R}_+^d} |f(x)| \prod_{j=1}^d \frac{1}{x_j}\,dx < \infty.$$

Theorem D and results from [7] are immediate corollaries of Theorem F. And, of course, Theorem E can easily be deduced, say, from Corollary 2.1.

3. Proofs

We give, step by step, proofs of the main results formulated in Introduction.

3.1. **Proof of Theorem 1.1.** To prove **a)**, we apply the first part of Theorem C.

Denoting $h(p) = \pi 2^{-p}$, $p \in \mathbb{Z}$, and

$$\Delta(h) = \left(\int_{\mathbb{R}} |f(t+h) - f(t-h)|^2 dt \right)^{1/2}, \tag{3.1}$$

we are going to prove that

$$\sum_{p=0}^{\infty} 2^{p/2} \Delta(h(p)) + \sum_{p=1}^{\infty} 2^{-p/2} \Delta(h(-p)) < \infty. \tag{3.2}$$

It is obvious that for $h > 0$

$$|f(t+h) - f(t-h)| \le 2 f_0(\min |t \pm h|), \tag{3.3}$$

$$|f(t+h) - f(t-h)| = |\int_{t-h}^{t+h} f'(s)\, ds| \le 2h f_1(\min |t \pm h|), \tag{3.4}$$

and for $|t| \le 3h$

$$|f(t+h) - f(t-h)| \le \int_{t-h}^{t+h} f_1(s)\, ds| \le 2 \int_0^{4h} f_1(t)\, dt. \tag{3.5}$$

The proof will be divided into several steps.

Step 1. To separately study the behavior near the origin and near infinity, we represent f as the sum of two functions φ and ψ with similar properties. First, let $\varphi(t) = f(t)$ when $|t| \le 2\pi$, while for $|t| \ge 2\pi$ it is $f(t)(3 - \frac{|t|}{\pi})_+$. Consequently, $\psi(t) = f(t) - \varphi(t)$.

Monotone majorants of the absolute values of these functions and their derivatives satisfy the inequalities

$$\varphi_0(t) \le f_0(t), \quad \varphi_1(t) \le f_1(t) + \frac{1}{\pi} f_0(t), \tag{3.6}$$

and

$$\psi_0(t) \le f_0(t), \quad \psi_1(t) \le \begin{cases} f_1(2\pi) + \frac{1}{\pi} f_0(2\pi), & |t| \le 3\pi, \\ f_1(t), & |t| \ge 3\pi. \end{cases} \tag{3.7}$$

Step 2. For the compactly supported function φ the second sum in (3.2) does not exceed

$$2\sum_{p=1}^{\infty} 2^{-p/2} ||\varphi||_2 \le 2\sum_{p=1}^{\infty} 2^{-p/2} \biggl(\int\limits_{-3\pi}^{3\pi} |f(t)|^2 dt \biggr)^{1/2} \le c_1 f_0(0).$$

Further,

$$\begin{aligned} f_0(0) &= \sup_{t\in\mathbb{R}} |f(t)| \le \sup_{|t|\le 2} |f(t)| + f_0(2) \\ &\le \sup_{|t|\le 2} |f(t) - f(2\mathrm{sign} t)| + 2f_0(2) \\ &\le \int\limits_0^2 f_1(t)\,dt + 2f_0(2) \le 2\int\limits_0^1 f_1(t)\,dt + 2f_0(2) \\ &\le \frac{2}{\ln 2}\int\limits_0^1 f_1(t) \ln\frac{2}{t}\,dt + \frac{2}{\ln 2}\int\limits_1^2 \frac{f_0(t)}{t}\,dt \le \frac{2}{\ln 2}(A_1 + A_0). \end{aligned} \tag{3.8}$$

Step 3. To estimate the first sum in (3.2), for φ equal to

$$\sum_{p=0}^{\infty} 2^{p/2} \biggl(\int\limits_{|t|\le 3\pi + h(p)} |\varphi(t + h(p)) - \varphi(t - h(p)|^2 dt \biggr)^{1/2},$$

we split the integral in (3.1) into the two ones: over $|t| \le 3h(p)$ and over $3h(p) \le |t|$.

Step 3.1. Using (3.5), we obtain for the first part

$$\begin{aligned}
&\sum_{p=0}^{\infty} 2^{p/2}\Bigg(\int\limits_{|t|\le 3h(p)} |\varphi(t+h(p)) - \varphi(t-h(p)|^2 dt\Bigg)^{1/2} \\
\le\;& 2\sum_{p=0}^{\infty} 2^{p/2}\Bigg(\int\limits_{-3h(p)}^{3h(p)} \Bigg(\int\limits_{0}^{4h(p)} \varphi_1(t)\,dt\Bigg)^2 du\Bigg)^{1/2} \\
=\;& 2\sqrt{6\pi}\sum_{p=0}^{\infty} \int\limits_{0}^{4h(p)} \varphi_1(t)\,dt.
\end{aligned}$$

We will systematically need to pass from sums to integrals. In the following simple inequalities one can pass to the limit as $m \to \infty$ and $n \to \infty$.

If g increases on $(0,\infty)$, and $n \ge m$, then for any $\alpha \in \mathbb{R}$

$$\begin{aligned}
\sum_{p=m}^{n} 2^{p\alpha} g(2^p) \;&\le\; \sum_{p=m}^{n} \int\limits_{2^p}^{2^{p+1}} t^{\alpha-1} g(t)\,dt \Bigg(\int\limits_{2^p}^{2^{p+1}} t^{\alpha-1}\,dt\Bigg)^{-1} \\
&=\; \frac{\alpha}{2^\alpha - 1} \int\limits_{2^m}^{2^{n+1}} t^{\alpha-1}\varphi(t)\,dt.
\end{aligned} \tag{3.9}$$

And if g decreases on $(0,\infty)$, and $n \ge m$, then for any $\alpha \in \mathbb{R}$

$$\begin{aligned}
\sum_{p=m}^{n} 2^{p\alpha} g(2^p) \;&\le\; \sum_{p=m}^{n} \int\limits_{2^{p-1}}^{2^{p}} t^{\alpha-1} g(t)\,dt \Bigg(\int\limits_{2^{p-1}}^{2^{p}} t^{\alpha-1}\,dt\Bigg)^{-1} \\
&=\; \frac{2^\alpha\alpha}{2^\alpha - 1} \int\limits_{2^{m-1}}^{2^{n}} t^{\alpha-1} g(t)\,dt.
\end{aligned} \tag{3.10}$$

Passing to the integral, changing the order of integration and substitution $s = 8\pi u$, we obtain, times a constant,

$$\int_0^{8\pi}\Big(\int_0^t \varphi_1(s)\,ds\Big)\,\frac{dt}{t}=\int_0^{8\pi}\varphi_1(u)\,du\int_u^{8\pi}\frac{dt}{t}$$

$$= \int_0^{8\pi}\varphi_1(u)\ln\frac{8\pi}{u}\,du=8\pi\int_0^1\varphi_1(8\pi u)\ln\frac{1}{u}\,du\le 8\pi A_1. \tag{3.11}$$

Step 3.2. Further, the second part of the sum does not exceed, by (3.4),

$$\sum_{p=0}^{\infty}2^{p/2}\left(\int\limits_{3h(p)\le|t|\le4\pi}|\varphi(t+h(p))-\varphi(t-h(p))|^2dt\right)^{1/2}$$

$$\le\ 2\sum_{p=0}^{\infty}2^{p/2}h(p)\left(\int\limits_{3h(p)\le|t|\le4\pi}\varphi_1^2(\min|t\pm h(p)|)\,dt\right)^{1/2}.$$

It follows from this, by taking into account the evenness of φ_1, the inequality (3.9) and the shift in the integral $u=s-h(p)$, that

$$4\pi\sum_{p=0}^{\infty}2^{-p/2}\left(\int_{\pi2^{-p}}^{4\pi}\varphi_1^2(s)\,ds\right)^{1/2}\le c_2\int_0^{4\pi}t^{-1/2}\left(\int_u^{4\pi}\varphi_1^2(u)\,du\right)^{1/2}dt$$

$$=\ c_2\int_0^{4\pi}\frac{1}{t^{1/2}\ln(8\pi/t)}\,\ln(8\pi/t)\left(\int_t^{4\pi}\varphi_1^2(u)\,du\right)^{1/2}dt.$$

Applying the Cauchy-Schwarz-Bunyakovskii inequality, we estimate the above through

$$
\begin{aligned}
& c_3 \left(\int_0^{4\pi} \ln^2 \frac{8\pi}{t} \int_t^{4\pi} \varphi_1^2(u)\, du\, dt \right)^{1/2} \\
= \;& c_3 \left(\int_0^{4\pi} \varphi_1^2(u) \int_0^{u} \ln^2 \frac{8\pi}{t}\, dt\, du \right)^{1/2} \\
\le \;& 30c_3 \left(\int_0^{4\pi} \varphi_1^2(t) t \ln^2(8\pi/t)\, dt \right)^{1/2} \\
\le \;& c_4 \left(\int_0^{1} \varphi_1^2(t) t \ln^2(2/t)\, dt \right)^{1/2} .
\end{aligned}
$$

Here, as above while establishing (3.11), monotonicity of φ_1 is used. Again by this, along with (3.6) and (3.8), we have

$$
\varphi_1(t)\ln(2/t) \le 2\int_{t/2}^{t} \varphi_1(u)\ln(2/u)\, du \le 2A_1 + c_5 f_0(0) \le c_6(A_0 + A_1).
$$

Therefore,

$$
\begin{aligned}
\left(\int_0^1 \varphi_1^2(t) t \ln^2(2/t)\, dt \right)^{1/2} &\le \sqrt{c_6(A_0+A_1)} \left(\int_0^1 \varphi_1(t)\ln(2/t)\, dt \right)^{1/2} \\
&\le \sqrt{c_6(A_0+A_1)}\sqrt{A_1 + c_7 f_0(0)} \le c_8(A_0 + A_1).
\end{aligned}
$$

Step 4. Let us proceed to the function $\psi(t) = f(t) - \varphi(t)$. It vanishes for $|t| \le 2\pi$ and coincides with f for $|t| \ge 3\pi$. Indeed, for $|t| \le 3\pi$ we have $\psi(t) = f(t)(\frac{|t|}{\pi} - 2)_+$; see also (3.7).

Step 4.1. To prove the validity of (3.2), let us start with the second sum. We split the integral in (3.1) into the two ones: over $|t| \le 3h(-p)$ and over $3h(-p) \le |t|$. We have (see (3.3) and (3.4))

$$\begin{aligned}
&\sum_{p=1}^{\infty}2^{-p/2}\Bigg(\int\limits_{3h(-p)\le|t|}|\psi(t+h(-p))-\psi(t-h(-p))|^2dt\Bigg)^{1/2}\\
\le\;& 2\sum_{p=1}^{\infty}2^{-p/2}\Bigg(\int\limits_{3h(-p)\le|t|}h(-p)\\
\times\;& \psi_0(\min|t\pm h(-p)|)\psi_1(\min|t\pm h(-p)|)\,dt\Bigg)^{1/2}\\
=\;& 2\sqrt{2\pi}\sum_{p=1}^{\infty}\Bigg(\int\limits_{3h(-p)}^{\infty}\psi_0(t-h(-p))\psi_1(t-h(-p))\,dt\Bigg)^{1/2}.
\end{aligned}$$

The last integral is

$$\begin{aligned}
& 2\sqrt{2\pi}\sum_{p=1}^{\infty}\Bigg(\int\limits_{2\pi 2^p}^{\infty}\psi_0(t)\psi_1(t)\,dt\Bigg)^{1/2}\\
\le\;& c_9\int\limits_1^{\infty}\Bigg(\int\limits_u^{\infty}\psi_0(t)\psi_1(t)\,dt\Bigg)^{1/2}\frac{du}{u}=c_9A_{0,1}.
\end{aligned}\tag{3.12}$$

Step 4.2. The rest of the second sum is

$$\begin{aligned}
&\sum_{p=1}^{\infty}2^{-p/2}\Bigg(\int\limits_{3h(-p)\ge|t|}|\psi(t+h(-p))-\psi(t-h(-p))|^2dt\Bigg)^{1/2}\\
\le\;& \sum_{p=1}^{\infty}2^{-p/2}\Bigg[\Bigg(\int\limits_{3h(-p)\ge|t|}|\psi(t+h(-p))|^2dt\Bigg)^{1/2}\\
+\;& \Bigg(\int\limits_{3h(-p)\ge|t|}|\psi(t-h(-p))|^2dt\Bigg)^{1/2}\Bigg]\\
\le\;& 2\sum_{p=1}^{\infty}2^{-p/2}\Bigg(\int\limits_{4h(-p)\ge|t|}|\psi(t)|^2dt\Bigg)^{1/2}.
\end{aligned}$$

The right-hand side does not exceed, times a constant,

$$\int_{2\pi}^{\infty}\Big(\int_{|t|\le u}|\psi(t)|^2dt\Big)^{1/2}\frac{du}{u^{3/2}}\le 2\int_{2\pi}^{\infty}\Big(\int_{2\pi}^{u}\psi_0(t)^2dt\Big)^{1/2}\frac{du}{u^{3/2}}. \tag{3.13}$$

We can consider in the sequel $\psi_0(2\pi)>0$, since otherwise, if $\psi_0(2\pi)=0$ (or, equivalently, if $\psi_1(2\pi)=0$) we have $\psi(t)\equiv 0$.

Integrating by parts in the last integral, we obtain

$$\Big[-\frac{2}{\sqrt{t}}\Big(\int_{2\pi}^{t}\psi_0^2(s)\,ds\Big)^{1/2}\Big]_{2\pi}^{\infty} + \int_{2\pi}^{\infty}\Big(\int_{2\pi}^{t}\psi_0^2(s)\,ds\Big)^{-1/2}\psi_0^2(t)\frac{dt}{\sqrt{t}}.$$

Integrated terms vanish at infinity by L'Hospital rule, say. For $t>2\pi$, by monotonicity of ψ_0,

$$\Big(\int_{2\pi}^{u}\psi_0^2(s)\,ds\Big)^{-1/2}\le\frac{1}{\psi_0(u)\sqrt{u-2\pi}}.$$

Since $u-2\pi\ge u/3$ when $u\ge 3\pi$, we arrive at the upper bound

$$\int_{2\pi}^{\infty}\frac{\psi_0(u)}{\sqrt{u(u-2\pi)}}\,du\le\psi_0(2\pi)\int_{2\pi}^{3\pi}\frac{du}{\sqrt{u(u-2\pi)}}+\sqrt{3}\int_{3\pi}^{\infty}\frac{\psi_0(u)}{u}\,du.$$

It remains to take into account that

$$\psi_0(2\pi)\le 2\int_{\pi}^{2\pi}\frac{\psi_0(t)}{t}\,dt\le 2A_0,$$

since also $\psi_0(t)\le f_0(t)$.

Step 4.3. Let us go on to the first sum in (3.2). Since $f(t)=0$ for $|t|\le 2\pi$, it is equal, with $h(p)=\pi 2^{-p}\in(0,\pi]$, to

$$\sum_{p=0}^{\infty} 2^{p/2}\left(\int\limits_{\max|t\pm h(p)|\ge 2\pi} |\psi(t+h(p))-\psi(t-h(p))|^2 dt\right)^{1/2}$$

$$(3.14)\quad = \sum_{p=0}^{\infty} 2^{p/2}\left(\int\limits_{|t|\ge 2\pi+h(p)} |\psi(t+h(p))-\psi(t-h(p))|^2 dt\right)^{1/2}.$$

Here we again split the integral into two, but in a different way: over $|t|\le 10\sqrt{2^p}$ and over $|t|\ge 10\sqrt{2^p}$. In the first one, we apply (3.4):

$$\sum_{p=0}^{\infty} 2^{p/2}\left(\int\limits_{2\pi+h(p)\le|t|\le 10\sqrt{2^p}} \psi_1^2(\min|t\pm h(p)|)\,dt\right)^{1/2} 2h(p)$$

$$\le\quad c_{10}\sum_{p=0}^{\infty} 2^{-p/2}\left(\int\limits_{2\pi}^{2\sqrt{\pi 2^p}} \psi_1^2(t)\,dt\right)^{1/2}.$$

Passing to the integral, we get (cf. (3.13))

$$\int\limits_{2\pi}^{\infty}\left(\int\limits_{2\pi}^{\sqrt{u}}\psi_1(t)^2dt\right)^{1/2}\frac{du}{u^{3/2}}.$$

Substituting $\sqrt{u}\to u$, we have

$$\int\limits_{2\pi}^{\infty}\left(\int\limits_{2\pi}^{u}\psi_1(t)^2dt\right)^{1/2}\frac{du}{u^2}.$$

Repeating similar estimations as for ψ_0 (see (3.13) and further), we obtain, times a constant,

$$\psi_1(2\pi)+\int\limits_{3\pi}^{\infty}\frac{\psi_1(t)}{t^{3/2}}\,dt\le c_{11}\psi_1(2\pi)\le c_{11}(f_1(2\pi)+f_0(2\pi)).$$

By monotonicity of f_0 and f_1 the last bound is obviously controlled by A_0+A_1 (see (3.8)).

Step 4.4. In the remained sum all is similar to getting (3.12). The only difference is that we get

$$\int_1^\infty \Big(\int_{\sqrt{u}}^\infty \psi_0(t)\psi_1(t)\,dt \Big)^{1/2} \frac{du}{u}$$

as an upper bound. But after substituting $\sqrt{u} \to u$, we obtain exactly $A_{0,1}$, which completes the proof of **a)**.

The proof of **b)** is also based on the first part of Theorem C. We first replace (3.4) with

$$(3.15)\ |f(t+h) - f(t-h)| = |\int_{t-h}^{t+h} f'(s)\,ds| \le 2hf_\infty(\max|t \pm h|).$$

It follows from $A_\delta < \infty$ that for $|t| \ge 2\pi$ and for the same $\delta \in (0,1)$

$$|f(t)| \le |f_0(t)| \le \left(\frac{A_\delta^{1+\delta}}{f_\infty(4\pi)}\right)^{1/\delta} |t|^{-1/\delta},$$

therefore

$$||f||_2^2 = \int_{|t|\ge 2\pi} |f(t)|^2 dt \le c_{12} \left(\frac{A_\delta^{1+\delta}}{f_\infty(4\pi)}\right)^{2/\delta}.$$

Hence the second sum in (3.2) does not exceed

$$(3.16) \qquad 2||f||_2 \sum_{p=1}^\infty 2^{-p/2} \le c_{13} \left(\frac{A_\delta^{1+\delta}}{f_\infty(4\pi)}\right)^{1/\delta}.$$

Let $\delta_1 = \frac{1-\delta}{1+\delta}$. Clearly, $\delta_1 \in (0,1)$. Applying to the first sum in (3.2) (cf. also (3.14)) simultaneously (3.3) and (3.15), we bound it with

$$
\begin{aligned}
& 2\sum_{p=0}^{\infty} 2^{p/2}\Bigg(\int\limits_{|t|\ge 2\pi+h(p)} f_0^{1-\delta_1}(\min|t\pm h(p)|) \\
\times \quad & h(p)^{1+\delta_1} f_\infty^{1+\delta_1}(\max|t\pm h(p)|)\,dt\Bigg)^{1/2} \\
= \quad & 4\sum_{p=0}^{\infty} 2^{p/2} h(p)^{(1+\delta_1)/2}\left(\int\limits_{2\pi}^{\infty} f_0^{1-\delta_1}(t) f_\infty^{1+\delta_1}(t+2h(p))\,dt\right)^{1/2} \\
\le \quad & \gamma_1(\delta)\left(\int\limits_{2\pi}^{\infty} f_0^{1-\delta_1}(t) f_\infty^{1+\delta_1}(t+2\pi)\,dt\right)^{1/2}.
\end{aligned}
$$

The choice of δ_1 and assumptions of the theorem yield for $t \ge 2\pi$

$$
f_0^{1-\delta_1}(t) f_\infty^{1+\delta_1}(t+2\pi) = (f_0^{\delta}(t) f_\infty(t+2\pi))^{2/(1+\delta)} \le \frac{A_\delta^2}{t^{\frac{2}{1+\delta}}},
$$

and the first sum does not exceed $\gamma_2(\delta)A_\delta$. Combining it with (3.16) gives the desired estimate.

It is also worth mentioning that $\max\limits_{2\pi\le|t|\le 4\pi} |f(t)| \le 2\pi f_\infty(4\pi)$.

The proof of the theorem is complete. □

3.2. **Proof of Corollary 1.2.** If $\alpha > 0$ and $\beta \ge 0$ with $\alpha+\beta > 1$, we apply **a)** of Theorem 1.1, while if $\beta < 0$ but still $\alpha+\beta > 1$, the assertion **b)** of Theorem 1.1 is applicable (one can take any δ satisfying $\delta\alpha+\beta = 1$).

There is a counterexample in the multiple case as well ([14, 7.4]). For $d = 1$, the function

$$
g(x) = \frac{e^{i|t|^{\alpha_1}}}{(1+|t|^2)^{\beta_1}}
$$

does not belong to $A(\mathbb{R})$ if $\alpha_1 \ne 1$ and $4\beta_1 < \alpha_1$.

If $\alpha \ne \beta$ we set $\beta_1 = \frac{\alpha}{2}$ and $\alpha_1 = \alpha - \beta + 1$. Then $4\beta_1 - \alpha_1 = \alpha + \beta - 1 < 0$ and for $|t| \to \infty$

$$
g(t) = O\Big(\frac{1}{|t|^\alpha}\Big), \quad g'(t) = O\Big(\frac{1}{|t|^\beta)}\Big).
$$

If $\alpha = \beta$ and $\alpha+\beta < 1$, one can a larger α so that the sum to still be less than 1 and then to make use of the above argument.

3.3. **Proof of Theorem 1.3.** The proof will go along the same lines, or, more precisely, the same steps, as that of **a)** of Theorem 1.1 does. We first represent the given function f as the sum of two functions φ and ψ so that φ to be compactly supported and near the origin coincide with f, while ψ correspondingly vanishes near the origin and coincides with f near infinity. Thus, let $\varphi(x) = f(x)$ when $|x_j| \le 2\pi$, $j = 1, 2, ..., d$. Further, when $|x_j| \ge 2\pi$ and $|x_j| \ge |x_k|$ for all $k = 1, 2, ..., d$, $\varphi(x) = f(x)(3 - \frac{|x_j|}{\pi})_+$, and this is for each $j = 1, 2, ..., d$. Correspondingly, $\psi(x) = f(x) - \varphi(x)$.

Extensions of (3.3), (3.4) and (3.5) are as follows. Let $r = \chi_1 + ... + \chi_d$, $h = (h_1, ..., h_d)$, and $x_{\chi,\pm h}$ be the vector x with x_j replaced by $|x_j \pm h_j|$ for $j : \chi_j = 1$.

$$|\prod_{j:\chi_j=1} \Delta_{h_j} f(x)| \le 2^r f_{\chi,\mathbf{0}}(\min x_{\chi,\pm h}), \tag{3.17}$$

$$|\prod_{j:\chi_j=1} \Delta_{h_j} f(x)| \le 2^r f_{\mathbf{0},\chi}(\min x_{\chi,\pm h}), \tag{3.18}$$

and when $|x_j| \le 3h_j$, $j : \chi_j = 1$

$$|\prod_{j:\chi_j=1} \Delta_{h_j} f(x)| \le 2^r \int\limits_{\substack{\prod[0,4h_j],\\ j:\chi_j=1}} f_{\mathbf{0},\chi}(x)\, dx_\chi. \tag{3.19}$$

Now, the result will follow from the next two propositions, in which φ and ψ are treated separately. The first one corresponds to the case in (1.4) when $\chi = \eta = \mathbf{0}$ and $\zeta = \mathbf{1}$.

Proposition 3.1. *Let $\varphi(x) \in C_0(\mathbb{R}^d)$ be supported on $\{x : |x_j| \le 3\pi, j = 1, ..., d\}$. Let φ and its partial derivatives $D^\eta\varphi$, $\mathbf{0} \le \eta < \mathbf{1}$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. If*

$$\int_0^1 \ln\frac{2}{x_1} \dots \int_0^1 \ln\frac{2}{x_d} |\varphi_{\mathbf{1}}(x)|\, dx < \infty, \tag{3.20}$$

then $\varphi \in A(\mathbb{R}^d)$.

Proof. We again use a) of Theorem C. By this, we have to estimate the sums of the form

$$\sum_{\substack{1\le s_i<\infty,\\ i:\chi_i=0}} 2^{-\frac12\sum\limits_{i:\chi_i=0} s_i} \sum_{\substack{0\le s_j<\infty,\\ j:\chi_j=1}} 2^{\frac12\sum\limits_{j:\chi_j=1} s_j} ||\Delta_{H(s)}(\varphi)||_2<\infty, \tag{3.21}$$

where $H(s)$ is the d-dimensional vector with the entries $\pi 2^{s_i}$ when $\chi_i=0$ and $\pi 2^{-s_j}$ when $\chi_j=1$. Proceeding to the first sum, we have

$$\begin{aligned}
&\sum_{\substack{1\le s_i<\infty,\\ i:\chi_i=0}} 2^{-\frac12\sum\limits_{i:\chi_i=0} s_i} ||\Delta_{H(s)}(\varphi)||_2\\
\le\ & 2^d \sum_{\substack{1\le s_i<\infty,\\ i:\chi_i=0}} 2^{-\frac12\sum\limits_{i:\chi_i=0} s_i} ||\prod_{j:\chi_j=1}\Delta_{\pi 2^{-s_j}}\varphi||_2\\
\le\ & 2^d \sum_{\substack{1\le s_i<\infty,\\ i:\chi_i=0}} 2^{-\frac12\sum_{i:\chi_i=0} s_i}\Bigg(\int\limits_{\substack{|x_i|\le 3\pi,\\ i:\chi_i=0}}\ \int\limits_{\substack{|x_i|\le 3\pi,\\ i:\chi_i=1}} |\prod_{j:\chi_j=1}\Delta_{\pi 2^{-s_j}} f(x)|^2dx\Bigg)^{1/2}\\
\le\ & c_{14}\Bigg(\int\limits_{\substack{|x_i|\le 3\pi,\\ i:\chi_i=1}}\Bigg(\prod_{j:\chi_j=1}\Delta_{\pi 2^{-s_j}} f\Bigg)_{\mathbf{1}-\chi,\mathbf{0}}(x_\chi^0)^2dx_\chi\Bigg)^{1/2},
\end{aligned}$$

where x_χ^0 is x with zero entries in place of x_i when $\chi_i=0$.

Going on to the second sum in (3.21), we have to estimate

$$\sum_{\substack{0\le s_j<\infty,\\ j:\chi_j=1}} 2^{\frac12\sum\limits_{j:\chi_j=1} s_j}\left(\int\limits_{\substack{|x_i|\le 3\pi,\\ i:\chi_i=1}}\Bigg(\prod_{j:\chi_j=1}\Delta_{\pi 2^{-s_j}} f\Bigg)_{\mathbf{1}-\chi,\mathbf{0}}(x_\chi^0)^2dx_\chi\right)^{1/2}.$$

But this is estimated similarly to the one-dimensional case, with calculations repeated in each j-th variable for $j:\chi_j=1$. This completes the proof. $\square$

Let us go on to ψ.

Proposition 3.2. *Let $\psi(x)\in C_0(\mathbb{R}^d)$ vanish on $\{x:|x_j|\le 2\pi, j=1,...,d\}$. Let ψ and its partial derivatives $D^\eta\psi$, $\mathbf{0}\le\eta<\mathbf{1}$, be locally absolutely continuous on $(\mathbb{R}\setminus\{0\})^d$ in each variable. Let also partial derivatives $D^\eta\psi$, $\mathbf{0}<\eta\le\mathbf{1}$ be almost everywhere bounded out of any neighborhood of each coordinate hyperplane. If*

$$\Big\| \int_1^\infty \dots \int_1^\infty \Big(\int_{\prod_{j:\eta_j=1}[u_j,\infty)} f_{\chi+\eta,\zeta}(x) f_{\chi,\eta+\zeta}(x)\, dx_\eta \Big)^{1/2} \prod_{\substack{i:\chi_i=1 \\ \text{or } \eta_i=1}} \frac{du_i}{u_i} \Big\|_{L_\infty(\mathbb{R}_\zeta)} < \infty$$

for all χ, η *and* ζ *such that* $\chi+\eta+\zeta = \mathbf{1} = \{1, 1, ..., 1\}$, *then* $f \in A(\mathbb{R}^d)$.

Proof. We again use **a)** of Theorem C. The steps of the proof will be similar to the four sub-steps of Step 4 in the one-dimensional proof. The first two are concerned with the sums where $1 \le s_j < \infty$ and the factor $2^{s_j/2}$ stays before the L^2 norm, while in the next two $0 \le s_j < \infty$ and the factor $2^{-s_j/2}$ stays before the L^2 norm. Each two steps correspond to splitting the relevant integrals over $|x_j| \le 3h(-s_j)$ and $3h(-s_j) \le |x_j|$ and $|x_j| \le \sqrt{3h(-s_j)}$ and $\sqrt{3h(-s_j)} \le |x_j|$, respectively. For each case separately the estimates are similar, we just repeat the same calculations in each variable involved. Let us indicate certain points in this procedure.

First, (3.17), (3.18) and (3.19) are used instead of (3.3), (3.4) and (3.5), respectively.

Then, in the one-dimensional case the order of estimations was not important, since they were completely independent. Here all types of estimates can be applicable at once, each with respect to certain group of variables assigned by χ, η and ζ. We start with an analog of Step 4.2. Let us show how to deal with integrated terms while integrating by parts. Without loss of generality, we can consider

$$\int_{2\pi}^\infty \int_{2\pi}^\infty \left(\int_{2\pi}^x \int_{2\pi}^y F^2(s,t)\, ds\, dt \right)^{1/2} \frac{dx}{x^{3/2}} \frac{dy}{y^{3/2}} \tag{3.22}$$

to be a model case, with F bounded and vanishing at infinity. Integrating by parts in y, we obtain

$$\left[-\frac{2}{\sqrt{y}} \left(\int_{2\pi}^x \int_{2\pi}^y F^2(s,t)\, ds\, dt \right)^{1/2} \right]_{2\pi}^\infty$$
$$+ \int_{2\pi}^\infty \left(\int_{2\pi}^x \int_{2\pi}^y F^2(s,t)\, ds\, dt \right)^{-1/2} \left(\int_{2\pi}^x F^2(s,y)\, ds\, dt \right) \frac{dy}{y^{1/2}}.$$

Since F is bounded, we can pass to the limit under the integral sign while using the L'Hospital rule as above. The estimates y are exactly

the same as in dimension one, and then we just repeat these in x. By this, (3.22) is controlled by

$$F(2\pi,2\pi)+\int\limits_{2\pi}^{\infty}\frac{F(x,2\pi)}{x}\,dx+\int\limits_{2\pi}^{\infty}\frac{F(2\pi,y)}{y}\,dy+\int\limits_{2\pi}^{\infty}\int\limits_{2\pi}^{\infty}\frac{F(x,y)}{xy}\,dx\,dy.$$

Further, we fulfil an analog of Step 4.3 for the corresponding group of variables. And after that we make use of analogs of Steps 4.1 and 4.4 that are, in essence, the same. We finally arrive at the desired estimate with $A_{\chi,\eta,\zeta}$. Of course, for some choices of χ, η and ζ not all types of estimates to be involved; there are extremal cases when only one type of estimates, that is, all sums and all integrals are of one type for all the variables, is fulfilled.

The interplay of the steps is the main problem in the multivariate extension, in dimension one all the steps are independent. To make such an interplay more transparent, let us consider all the details with easier notation in dimension two. More precisely, we consider the cases where first Steps 4.1 and 4.2 and then Steps 4.1 and 4.4 occur simultaneously. Since both cases are two-dimensional we switch the notation from that with subscripts to the one with different letters.

Thus, let us estimate

$$\begin{aligned}
&\sum_{p=1}^{\infty}2^{-p/2}\sum_{q=1}^{\infty}2^{-q/2}\Bigg(\int\limits_{|s|\ge 3\pi 2^p}\int\limits_{|t|\le 3\pi 2^q}|f(s+h(-p),t+h(-q))\\
-\;\;&f(s-h(-p),t+h(-q))-f(s+h(-p),t-h(-q))\\
+\;\;&f(s-h(-p),t-h(-q))|^2ds\,dt\Bigg)^{1/2}.
\end{aligned}$$

Denoting, for brevity, $\Omega(s,t)=f(s+h(-p),t)f(s-h(-p),t)$, we first estimate, more or less along the same lines as in Step 4.2,

$$\sum_{q=1}^{\infty}2^{-q/2}\Bigg(\int\limits_{|t|\le 3\pi 2^q}\int\limits_{|s|\ge 3\pi 2^p}|\Omega(s,t+h(-q))-\Omega(s,t-h(-q))|^2ds\,dt\Bigg)^{1/2}$$

$$\leq \sum_{q=1}^{\infty} 2^{-q/2} \Big(\int\limits_{|t|\leq 3\pi 2^q} \int\limits_{|s|\geq 3\pi 2^p} |\Omega(s, t + h(-q))|^2 ds\, dt \Big)^{1/2}$$

$$+ \sum_{q=1}^{\infty} 2^{-q/2} \Big(\int\limits_{|t|\leq 3\pi 2^q} \int\limits_{|s|\geq 3\pi 2^p} |\Omega(s, t - h(-q))|^2 ds\, dt \Big)^{1/2}$$

$$\leq 2\sum_{q=1}^{\infty} 2^{-q/2} \Big(\int\limits_{|t|\leq 4\pi 2^q} \int\limits_{|s|\geq 3\pi 2^p} |\Omega(s, t)|^2 ds\, dt \Big)^{1/2}$$

$$\leq c_{15} \int_{2\pi}^{\infty} \Big(\int\limits_{|t|\leq y} \int\limits_{|s|\geq 3\pi 2^p} |\Omega(s, t)|^2 ds\, dt \Big)^{1/2} dy$$

$$\leq c_{16} \int_{2\pi}^{\infty} \Big(\int_{2\pi}^{y} \int\limits_{|s|\geq 3\pi 2^p} |\Omega_0(s, t)|^2 ds\, dt \Big)^{1/2} \frac{dy}{y^{3/2}}.$$

Here Ω_0 means, for a moment, the same as f_0 in Theorem 1.1 but with respect to one of the variables, t. Similarly, Ω_1 is an analog of f_1. Exactly as in the Step 4.2 we bound the right-hand side by

$$c_{17} \Big(\int_0^1 \Big(\int\limits_{|s|\geq 3\pi 2^p} |\Omega_1(s, y)|^2 ds \Big)^{1/2} \ln \frac{2}{y}\, dy$$

$$+ \int_1^{\infty} \Big(\int\limits_{|s|\geq 3\pi 2^p} |\Omega_0(s, y)|^2 ds \Big)^{1/2} \frac{dy}{y} \Big). \tag{3.23}$$

What remains is to estimate

$$\sum_{p=1}^{\infty} 2^{-p/2} \Big(\int\limits_{|s|\geq 3\pi 2^p} |F(s + h(-p), y) - F(s - h(-p), y)|^2 ds \Big)^{1/2},$$

where F denotes either Ω_0 or Ω_1. But this is exactly Step 4.1 above. Fulfilling it, we arrive at one of the relations of type (1.4), with

$$f_{(1,0),(0,1)} f_{(0,0),(1,1)}$$

inside while estimating the first summand in (3.23) and

$$f_{(1,1),(0,0)}f_{(0,1),(1,0)}$$

for the second one.

Let us go on to to the combination of Steps 4.1 and 4.4. We are proceeding to

$$\begin{aligned}\sum_{p=1}^{\infty}2^{-p/2}\sum_{q=0}^{\infty}2^{q/2}\Bigl(\int\limits_{|s|\ge 3\pi 2^p}\int\limits_{|t|\ge\sqrt{3\pi 2^q}}&|f(s+h(-p),t+h(-q))\\ -\quad & f(s-h(-p),t+h(-q))-f(s+h(-p),t-h(-q))\\ +\quad & f(s-h(-p),t-h(-q))|^2ds\,dt\Bigr)^{1/2}.\end{aligned}$$

With the same notation Ω in hand and using Step 4.4, we estimate

$$\sum_{q=0}^{\infty}2^{-q/2}\Bigl(\int\limits_{|t|\ge\sqrt{3\pi 2^q}}\int\limits_{|s|\ge 3\pi 2^p}|\Omega(s,t+h(-q))-\Omega(s,t-h(-q))|^2ds\,dt\Bigr)^{1/2}$$

$$\begin{aligned}&\le\quad c_{18}\sum_{q=0}^{\infty}\Bigl(\int\limits_{|s|\ge 3\pi 2^p}\int\limits_{\sqrt{3\pi 2^q}}^{\infty}\Omega_0(s,t)\Omega_1(s,t)\,dt\,ds\Bigr)^{1/2}\\ &\le\quad c_{19}\int\limits_{2\pi}^{\infty}\Bigl(\int\limits_{|s|\ge 3\pi 2^p}\int\limits_{y}^{\infty}\Omega_0(s,t)\Omega_1(s,t)\,dt\,ds\Bigr)^{1/2}\frac{dy}{y}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{p=1}^{\infty}2^{-p/2}\Bigl(\int\limits_{y}^{\infty}\int\limits_{\sqrt{3\pi 2^q}}^{\infty}\Omega_0(s,t)\Omega_1(s,t)\,ds\,dt\Bigr)^{1/2}\\ \le\quad & c_{20}\int\limits_{2\pi}^{\infty}\Bigl(\int\limits_{x}^{\infty}\int\limits_{y}^{\infty}f_{(1,1),(0,0)}(s,t)f_{(0,0),(1,1)}(s,t)\,ds\,dt\Bigr)^{1/2}\frac{dx}{x}\frac{dy}{y}.\end{aligned}$$

This is the needed bound. We just note that in the same way one could have in the inner integral $f_{(0,1),(1,0)}f_{(1,0),(0,1)}$ or even

$$\sqrt{f_{(1,1),(0,0)}f_{(0,1),(1,0)}f_{(1,0),(0,1)}f_{(0,0),(1,1)}}.$$

The proof is complete. □

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